

Part A – Opinion & Influence (2/4)

Manipulation

COMS 6998-1: Economic Dynamics in Social Networks
Wednesday, September 25th

Outline of Part A

- Intro: Life under the influence
- Consensus and Social Learning (Lec. 1)
- What's against consensus?
 - Manipulation: self-interested nodes (Lec. 2)
 - Bias: homophily, selection, polarization (Lec. 3)
- Influence for adoption, spread
 - How to model? How to leverage? (Lec. 4)

Manipulation (roadmap)

- So far, consensus from a fair averaging
 - And, under some conditions, approx. correct
- Not exactly what reality presents, and indeed
 - People may play the game **with a bias**
 - People may **not always** play the game
 - People may **never** play the game

Model type A: Nodal update

- A node makes a transition,
 - sees all available information, updates x_i
 - typically, using x_i and $(x_j)_{\{j \mid j \text{ in } N(i)\}}$ where $N(i) = \{j \mid i \text{ listens to } j\}$
 - Synchronous: all nodes update simultaneously
or Asynchronous: i wakes up at random times
- Possible variants
 - **Fair**: $x_i := f_i(x_i, (x_j)_{j \mid i \text{ listens to } j})$ (i.e., weight avg)
 - **Biased**: $x_i := f_i(x_i(0), (x_j)_{j \mid i \text{ listens to } j})$
 - **Stubborn**: $x_i := x_i$ (and, hence, $= x_i(0)$)

Some examples:

- Synchronous, fair,
 - Our previous work! Iterated algorithm
 - Cvgce (connected, aperiodic), spectral prop
 - $E = \{ (i,j) \mid j \text{ in } N(i) \text{ and } f_i \text{ puts weight on } x_j \}$ str. connected
- Synchronous, biased and/or stubborn
 - Does it converge? Connectedness? To what?
 - Can we characterize the quality of equilibria?
 - When are equilibria attained efficient?

Model type B: Pairwise update

- A directed pair (i,j) makes a **transition**
 - meet, exchange information, update x_i and x_j
 - asynchronous: transition occurs **randomly**
 - Continuous time with Poisson Process (i,j) for transitions
 - Equivalently, discrete time and pick **one** (i,j) w.p. $\frac{1}{n} p_{i,j}$
- Possible interaction
 - **Symmetric:** $x_i \leftarrow \frac{x_i + x_j}{2}$ and $x_j \leftarrow \frac{x_i + x_j}{2}$
 - **Asymmetric:** $x_i \leftarrow (1 - \theta_{i,j}) \cdot x_i + \theta_{i,j} \cdot x_j$
 - **Mixed:** weighted averaging,
generally, symmetric w.p. $\beta_{i,j}$ and asym. w.p. $\alpha_{i,j}$

Some examples:

- Only symmetric interaction ($\beta_{i,j}=1$) ?
 - Close to our previous work but distinct
 - E.g., cvgce to the **fair** average for any matrix P
 - Argument: mass conservation
 - E.g., $E=\{ (i,j) \text{ such that } p_{i,j} > 0 \}$ str. connected
- More generally, we say (i,j) **forceful** if $\alpha_{i,j}>0$
 - Is the system converging? Connectedness?
 - What is the impact of forceful links?

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Explaining the reason for bias

- Let us assume opinion = cost minimization

$$x_i \leftarrow \min_x \text{Cost}_i(x_i, (x_j)_{j \in \mathcal{N}(i)})$$

-E.g. $\text{Cost}_i = (x_i - x_i(0))^2 + \sum_{j \in \mathcal{N}(i)} w_{i,j} (x_i - x_j)^2$

-So that:
$$x_i = \frac{1}{1 + \sum_{j \in \mathcal{N}(i)} w_{i,j}} \left(x_i(0) + \sum_{j \in \mathcal{N}(i)} w_{i,j} x_j \right)$$

Explaining the reason for bias

- Let us assume opinion = cost minimization

$$x_i \leftarrow \min_x \text{Cost}_i(x_i, (x_j)_{j \in \mathcal{N}(i)})$$

-E.g. $\text{Cost}_i = K_i(x_i - x_i(0))^2 + \sum_{j \in \mathcal{N}(i)} (x_i - x_j)^2$

-So that:
$$x_i = \frac{1}{K_i + |\mathcal{N}(i)|} \left(K_i x_i(0) + \sum_{j \in \mathcal{N}(i)} x_j \right)$$

Rewriting the synchronous evolution

○ We have $x(t+1) = Ax(t) + Bx(0)$ where

$$A_{i,j} = \frac{1}{K_i + |N(i)|} \text{ for } j \in N(i), \text{ otherwise } = 0$$

$$B = \text{Diag}\left(\frac{K_i}{|N(i)| + K_i}\right)$$

$$x(t) = A^t x(0) + \sum_{l=0}^{t-1} A^l B x(0)$$

Convergence

○THM: Equilibrium exists

$$x(\infty) = \sum_{t=0}^{\infty} A^t Bx(0) = (I - A)^{-1} Bx(0)$$

- Assuming str. connectivity for $E = \{(i,j) \mid j \in N(i)\}$
- Assuming at least one agent is biased
- Proof: A is a sub-stochastic matrix
- Note it implies initial opinion of unbiased agents eventually disappear

J. Ghaderi and R. Srikant, "Opinion Dynamics in Social Networks: A Local Interaction Game with Stubborn Agents," ACC '13: Proceedings of the American Control Conference, vol. cs.GT. 2013.

Equilibrium

- Assume all biased are stubborn

for any i , either $K_i = 0$ or $K_i = \infty$

- THM: We can understand the equilibrium

$$x_i(\infty) = \sum_{j \in S} P[Z(\tau_S) = j \mid Z(0) = i] \cdot x_j ,$$

- Where Z is a simple random walk

- τ_S is the first time it hits the set of stubborn

- It generalizes for biased agents

Proof of equilibrium formula

○ We consider the case where nodes are either unbiased or stubborn

○ STEP1: Prove that

$$x(t+1) = \tilde{A}x(t) + \tilde{B}x^{(S)}(0)$$

– Where $x^{(S)}(0)$ is the vector stubborn nodes

– And \tilde{A} is obtained by removing stubborn nodes

○ STEP2: Deduce: $x(\infty) = \sum_{t \geq 0} \tilde{A}^t \tilde{B}x^{(S)}(0) = (I - \tilde{A})^{-1} \tilde{B}x^{(S)}(0)$

○ STEP3: Check that $F_{i,j} = P[Z(\tau_S) = j | Z(0) = i]$ satisfies $F = (I - \tilde{A})^{-1} \tilde{B}$

Efficiency of equilibrium

- Let x be a Nash equilibrium of biased game
 - How can x compares to
 - Examples:

- THM: For symmetric graphs
 - Equilibrium is at most $9/8$ more costly overall than the absolute best
 - Price of Anarchy is at most $9/8$

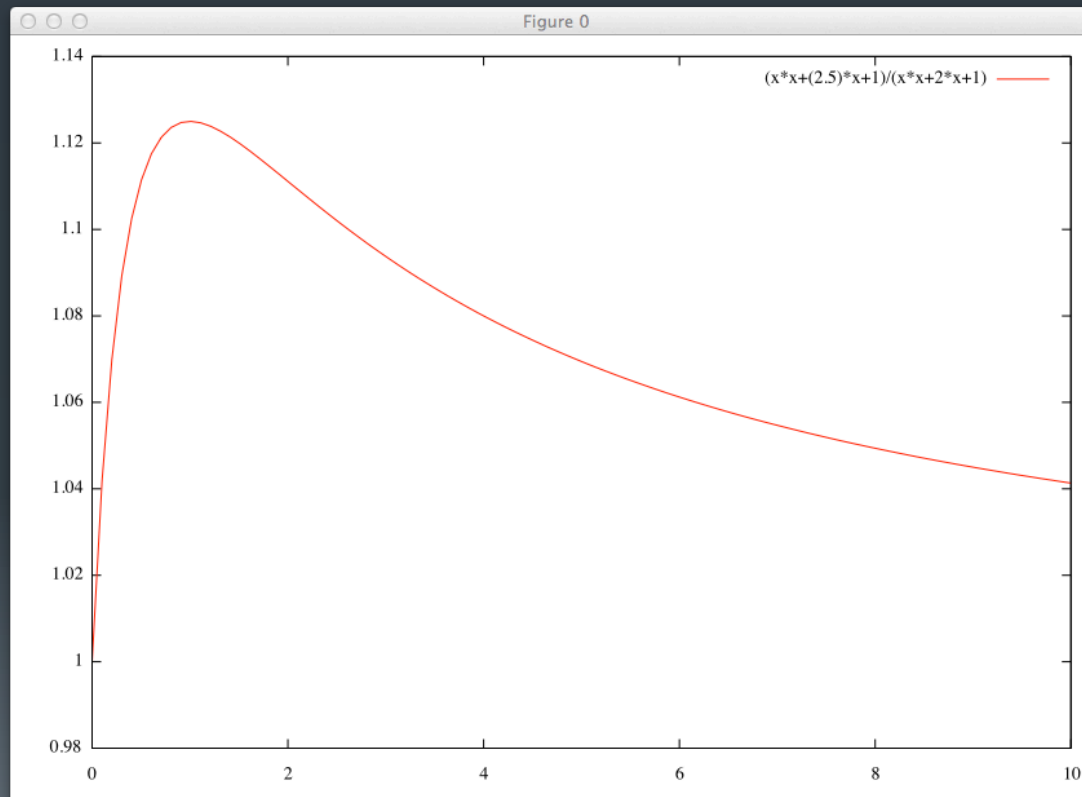
D. Bindel, J. M. Kleinberg, and S. Oren, “How Bad is Forming Your Own Opinion?” Foundations of Computer Science (FOCS), 2011 IEEE 52nd Annual Symposium on, pp. 57–66, 2011.

Proof of price of Anarchy

- Step 1: $\text{Cost}(x) = 2 \cdot x^T L x + \|x - x(0)\|^2$
 - Where L is the Laplacian matrix (Deg-Adjac)
- Step 2: The optimal cost is hence attained when $(2L + I)x^* = x(0)$, and the equilibrium of the system satisfies $(I + L)\tilde{x} = x(0)$
- Step 3: $I+L$ is positive definite. A change of base renders all these matrix diagonal.
- Step 4: The maximum ration between the cost is the max of a function taken in eigenvalues

Function in the proof

- The function of the ration has maximum $9/8$



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The two situations we consider

Case 1: Everyone indirectly influenced by all

- $E = \{ (i,j) \text{ active} \mid p_{i,j}(\beta_{i,j} + \alpha_{i,j}) > 0 \}$ st. connected
- Sometimes called “**nobody on an island**”

Case 2: All indirectly influenced by stubborn

- i called **stubborn** is $p_{i,j}(\beta_{i,j} + \alpha_{i,j}) = 0$ for all j
- x_i hence remains static

Does this cover all cases?

- Almost all of them by reduction

Type B revisited

○ Dynamics may be rewritten

– Where T is a random matrix $x^{t+1} = T^t x^t$

– Whose expectation satisfies $\tilde{T} = E[T^t] = C + D$

– With

$$C = \frac{1}{n} \sum_{i,j} p_{i,j} ((1 - \alpha_{i,j} - \beta_{i,j})I + (\alpha_{i,j} + \beta_{i,j})A^{i,j})$$

$$A^{i,j} = I - \frac{(e_i - e_j)(e_i - e_j)'}{2}$$

– And D is a **perturbation** from forceful links

$$D = \frac{1}{n} \sum_{i,j} p_{i,j} \alpha_{i,j} (J^{i,j} - A^{i,j}) \quad J_{i,j} = I - \theta_{i,j} e_i (e_i - e_j)'$$

Main result: Convergence

- Model type B, Case 1 “nobody on an island”
- THM1-2: All nodes converges to the same **random linear combination** x^∞

$$\forall i, \lim_{k \rightarrow \infty} x_i^t = x^\infty = \sum_j s_j x_i^0$$

weight s_j are **independent** of initial values x^0

$E[s_j]$ is weight of j associated with $E[T]$

- Forceful links create random perturbation

D. Acemoglu, A. Ozdaglar, and A. ParandehGheibi, “Spread of (mis) information in social networks,” *Games and Economic Behavior*, vol. 70, no. 2, pp. 194–227, 2010.

Main result: Perturbation

○ How non-uniform are weights $E[s]$?

○ THM6: let $\lambda_2(C)$ be the 2nd larg. eigenval. C

$$\|E[s] - \frac{1}{n}e\|_2 \leq \frac{1}{1 - \lambda_2(C)} \frac{\sum_{i,j} p_{i,j} \alpha_{i,j}}{n}$$

– As a consequence, a society with good mixing and few forceful agents will not deviate much

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A model of stubborn agents

- Assume agents are either regular or **stubborn**
- Slotted time dynamics
 - Pick node i , meets j with proba $p_{i,j}$
$$x_i^{t+1} = (1 - \theta_{i,j})x_i^t + \theta_{i,j}x_j^t$$
 - Stubborn simply do not move
 - Link (i,j) active if $p_{i,j}\theta_{i,j} > 0$
- Case 2: any regular agent i can reach a stubborn agents through active links.

Main results: Convergence

- THM1-3: The process of opinions converges in distribution to a stationary belief vector x .
 - x is given a fixed point of the evolution
 - $E[x_i]$ linear combination of stubborn opinions with coefficients satisfying Kolmogorov eq.
- Big difference: random limit that is not a consensus

D. Acemoglu, G. Como, F. Fagnani, and A. Ozdaglar, “Opinion fluctuations and disagreement in social networks,” *Mathematics of Operations Research*, vol. 38, no. 1, pp. 1–27, 2013.