Part A – Opinion & Influence (2/4)

Manipulation

COMS 6998-1: Economic Dynamics in Social Networks
Wednesday, September 25th

Outline of Part A

- Intro: Life under the influence

- Consensus and Social Learning (Lec. 1)
- What's against consensus?
  - Manipulation: self-interested nodes (Lec. 2)
  - Bias: homophily, selection, polarization (Lec. 3)
- Influence for adoption, spread
  - How to model? How to leverage? (Lec. 4)

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Manipulation (roadmap)

- So far, consensus from a fair averaging
  - And, under some conditions, approx. correct

- Not exactly what reality presents, and indeed
  - People may play the game with a bias
  - People may not always play the game
  - People may never play the game

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Model type A: Nodal update

- A node makes a transition,
  - sees all available information, updates xi
    - typically, using x_i and $(x_j)_{\{j \mid j \text{ in } N(i)\}}$ where N(i)={j | i listens to j}
  - Synchronous: all nodes update simultaneously

  or Asynchronous: i wakes up at random times
- Possible variants
  - Fair: $x_i := f_i(x_i, (x_j)_{j \mid I \text{ listens to } j})$ (i.e., weight avg)
  - Biased: $x_i := f_i(x_i(0), (x_j)_{j \mid I \text{ listens to } j})$
  - Stubborn: $x_i := x_i$ (and, hence, $= x_i(0)$)

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Some examples:

- Synchronous, fair,
  - Our previous work! Iterated algorithm
  - Cvgce (connected, aperiodic), spectral prop
    - E={ (i,j) | j in N(i) and f_i puts weight on x_j} str. connected

- Synchronous, biased and/or stubborn
  - Does it converge? Connectedness? To what?
  - Can we characterize the quality of equilibria?
  - When are equilibria attained efficient?

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Model type B: Pairwise update

- A directed pair (i,j) makes a transition
  - meet, exchange information, update x_i and x_j
  - asynchronous: transition occurs randomly
    - Continuous time with Poisson Process (i,j) for transitions
    - Equivalently, discrete time and pick one (i,j) w.p. $\frac{1}{n}p_{i,j}$
- Possible interaction
  - Symmetric: $x_i \leftarrow \frac{x_i+x_j}{2}$ and $x_j \leftarrow \frac{x_i+x_j}{2}$
  - Asymmetric: $x_i \leftarrow (1-\theta_{i,j})\cdot x_i + \theta_{i,j}\cdot x_j$
  - Mixed: weighted averaging,
    generally, symmetric w.p. $\beta_{i,j}$ and asym. w.p. $\alpha_{i,j}$

Some examples:

- Only symmetric interaction ($\beta_{i,j}$=1) ?
  - Close to our previous work but distinct
  - E.g., cvgce to the fair average for any matrix P
  - Argument: mass conservation
    - E.g., E={ (i,j) such that $p_{i,j}$ > 0 } str. connected
- More generally, we say (i,j) forceful if $\alpha_{i,j}$>0
  - Is the system converging? Connectedness?
  - What is the impact of forceful links?

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Explaining the reason for bias

- Let us assume opinion = cost minimization

$$x_i \leftarrow \min_x \mathrm{Cost}_i \left(x_i, (x_j)_{j\in\mathcal{N}(i)}\right)$$

– E.g. $\mathrm{Cost}_i = (x_i - x_i(0))^2 + \sum_{j\in N(i)} w_{i,j}(x_i - x_j)^2$

– So that: $x_i = \dfrac{1}{1+\sum_{j\in N(i)} w_{i,j}} \left(x_i(0) + \sum_{j\in N(i)} w_{i,j} x_j \right)$

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Explaining the reason for bias

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$$x_i \leftarrow \min_x \mathrm{Cost}_i\left(x_i, (x_j)_{j\in\mathcal{N}(i)}\right)$$

  - E.g. $\mathrm{Cost}_i = K_i(x_i - x_i(0))^2 + \sum_{j\in N(i)}(x_i - x_j)^2$
  - So that: $x_i = \dfrac{1}{K_i + |N(i)|}\left(K_i x_i(0) + \sum_{j\in N(i)} x_j\right)$

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Rewriting the synchronous evolution

- We have $x(t+1) = Ax(t) + Bx(0)$ where

$$A_{i,j} = \frac{1}{K_i + |N(i)|} \text{ for } j \in N(i), \text{ otherwise } = 0$$

$$B = \mathrm{Diag}(\frac{K_i}{|N(i)| + K_i})$$

$$x(t) = A^t x(0) + \sum_{l=0}^{t-1} A^l B x(0)$$

Convergence

- THM: Equilibrum exists

$$x(\infty) = \sum_{t=0}^{\infty} A^t Bx(0) = (I - A)^{-1} Bx(0)$$

  - Assuming str. connectivity for E={(i,j) | j in N(i)}
  - Assuming at least one agent is biased
  - Proof: A is a sub-stochastic matrix
  - Note it implies initial opinion of unbiased agents eventually disappear

J. Ghaderi and R. Srikant, “Opinion Dynamics in Social Networks: A Local Interaction Game with Stubborn Agents,” ACC '13: Proceedings of the American Control Conference, vol. cs.GT. 2013.

Equilibrium

- Assume all biased are stubborn

$$\text{for any } i, \text{ either } K_i = 0 \text{ or } K_i = \infty$$

- THM: We can understand the equilibrium

$$x_i(\infty) = \sum_{j \in S} P[Z(\tau_S) = j \mid Z(0) = i] \cdot x_j \ ,$$

  - Where Z is a simple random walk
  - τ_S is the first time it hits the set of stubborn
- It generalizes for biased agents

Proof of equilibrium formula

- We consider the case where nodes are either unbiased or stubborn
- STEP1: Prove that

$$x(t+1) = \tilde{A}x(t) + \tilde{B}x^{(S)}(0)$$

  - Where $x^{(S)}(0)$ is the vector stubborn nodes
  - And $\tilde{A}$ is obtained by removing stubborn nodes
- STEP2: Deduce: $x(\infty) = \sum_{t\geq 0} \tilde{A}^t \tilde{B}x^{(S)}(0) = (I-\tilde{A})^{-1}\tilde{B}x^{(S)}(0)$
- STEP3: Check that $F_{i,j} = P[Z(\tau_S) = j | Z(0) = i]$ satisfies $F = (I-\tilde{A})^{-1}\tilde{B}$

Efficiency of equilibrium

- Let x be a Nash equilibrium of biased game
  - How can x compares to
  - Examples:

- THM: For symmetric graphs
  - Equilibrium is at most 9/8 more costly overall than the absolute best
  - Price of Anarchy is at most 9/8

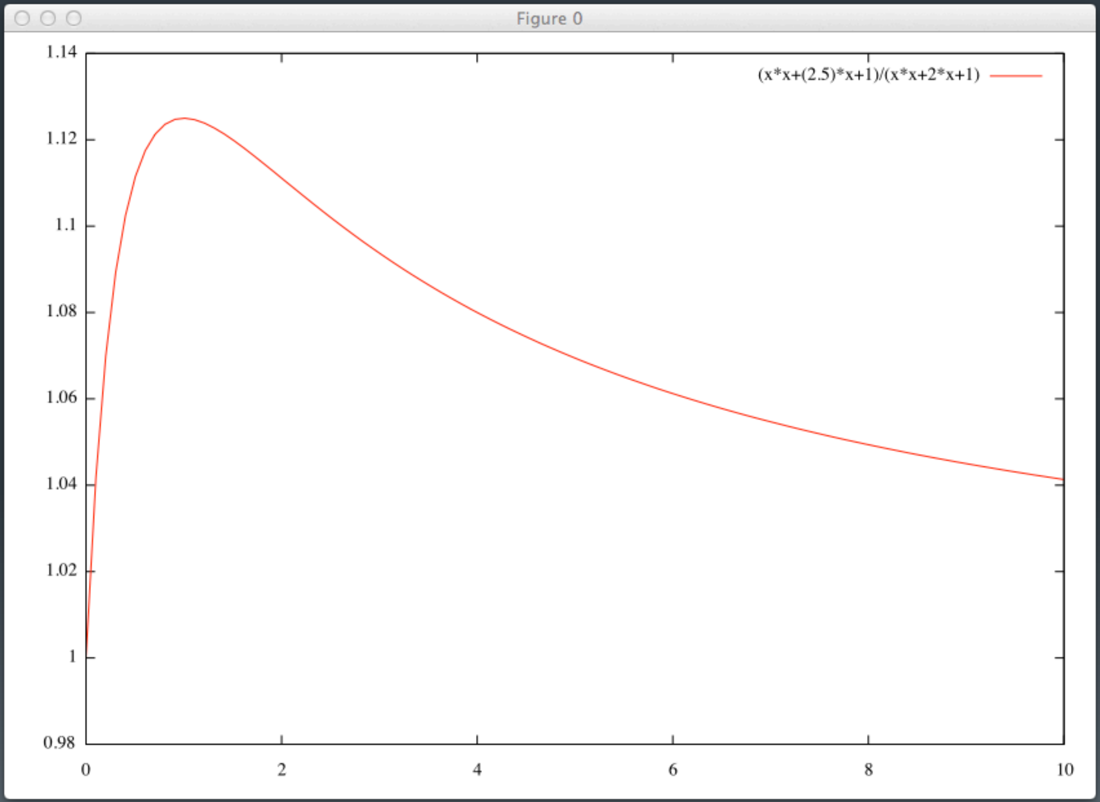
D. Bindel, J. M. Kleinberg, and S. Oren, “How Bad is Forming Your Own Opinion?,” Foundations of Computer Science (FOCS), 2011 IEEE 52nd Annual Symposium on, pp. 57–66, 2011.

Proof of price of Anarchy

- Step1: $\mathrm{Cost}(x) = 2 \cdot x^T L x + ||x - x(0)||^2$
  - Where L is the Laplacian matrix (Deg-Adjac)
- Step2: The optimal cost is hence attained when $(2L + I)x^* = x(0)$, and the equilibrium of the system satisfies $(I + L)\tilde{x} = x(0)$
- Step3: I+L is positive definite. A change of base renders all these matrix diagonal.
- Step4: The maximum ration between the cost is the max of a function taken in eigenvalues

Function in the proof

- The function of the ration has maximum 9/8



Manipulation (roadmap)

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  - People may *never* play the game

The two situations we consider

Case 1: Everyone indirectly influenced by all

- $E=\{ (i,j) \text{ active} \mid p_{i,j}(\beta_{i,j}+\alpha_{i,j})>0 \}$ st. connected
- Sometimes called "nobody on an island"

Case 2: All indirectly influenced by stubborns

- i called stubborn is $p_{i,j}(\beta_{i,j}+\alpha_{i,j}) = 0$ for all j
- x_i hence remains static

Does this cover all cases?

- Almost all of them by reduction

Type B revisited

- Dynamics may be rewritten
  - Where T is a random matrix $x^{t+1} = T^t x^t$
  - Whose expectation satisfies $\tilde{T} = E[T^t] = C + D$
  - With

$$C = \frac{1}{n}\sum_{i,j} p_{i,j}((1-\alpha_{i,j}-\beta_{i,j})I + (\alpha_{i,j}+\beta_{i,j})A^{i,j})$$

$$A^{i,j} = I - \frac{(e_i - e_j)(e_i - e_j)'}{2}$$

  - And D is a perturbation from forceful links

$$D = \frac{1}{n}\sum_{i,j} p_{i,j}\alpha_{i,j}(J^{i,j} - A^{i,j}) \qquad J_{i,j} = I - \theta_{i,j} e_i (e_i - e_j)'$$

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Main result: Convergence

- Model type B, Case 1 “nobody on an island”
- THM1-2: All nodes converges to the same random linear combination x^∞

$$\forall i\,,\ \lim_{k\to\infty} x_i^t = x^\infty = \sum_j s_j x_i^0$$

weight s_j are independent of initial values x^0

$E[s_j]$ is weight of j associated with E[T]

- Forceful links create random perturbation

D. Acemoglu, A. Ozdaglar, and A. ParandehGheibi, “Spread of (mis) information in social networks,” Games and Economic Behavior, vol. 70, no. 2, pp. 194–227, 2010.

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Main result: Perturbation

- How non-uniform are weights E[s]?

- THM6: let $\lambda_2(C)$ be the 2nd larg. eigenval. C

$$||E[s] - \frac{1}{n}e||_2 \leq \frac{1}{1-\lambda_2(C)} \frac{\sum_{i,j} p_{i,j}\alpha_{i,j}}{n}$$

  - As a consequence, a society with good mixing and few forceful agents will not deviate much

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A model of stubborn agents

- Assume agents are either regular or stubborn
- Slotted time dynamics
  - Pick node i, meets j with proba $p_{i,j}$

$$x_i^{t+1} = (1-\theta_{i,j})x_i^t + \theta_{i,j}x_j^t$$

  - Stubborn simply do not move
  - Link (i,j) active if $p_{i,j}\theta_{i,j} > 0$
- Case 2: any regular agent i can reach a stubborn agents through active links.

Main results: Convergence

- THM1-3: The process of opinions converges in distribution to a stationary belief vector x.
  - x is given a fixed point of the evolution
  - $E[x_i]$ linear combination of stubborn opinions with coefficients satisfying Kolmogorov eq.
- Big difference: random limit that is not a consensus

D. Acemoglu, G. Como, F. Fagnani, and A. Ozdaglar, "Opinion fluctuations and disagreement in social networks," Mathematics of Operations Research, vol. 38, no. 1, pp. 1–27, 2013.